ON SOME GENERAL THEOREMS CONCERNING ORDINARY CLOSED CURVES.*

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In an article which appeared in the Rendiconti del Circolo Matematico di Palermo† it is proved analytically that every closed analytic curve, without any other singularities than a finite number of inflexions, has an even number of inflexions and admits of an even number of tangents parallel to any given direction. The proofs of these theorems are based upon the coperiodicity of the parametric representations of such curves.

The method, these results and others, may be extended to any closed analytic curve without rectilinear segments and with a finite number of singularities of an algebraic character.

1. Theorems on Periodic Functions. To prove the more general theorems I shall make use of the following simple propositions on functions of a real variable:

THEOREM I. *Given a real periodic function* $u = F(t)$ *of the real variable* t *with the primitive period interval* $0 \leqq t < w$, *which is uniform and continuous for all values of* t *in the interval, except for a finite number* ν *of values* $t_1, t_2, t_3, \cdots, t_k, \cdots, t_\nu$, *for which* u *becomes infinite in such a manner that when* ϵ *is an arbitrarily small positive real number,*

$$\lim_{\epsilon = 0} F(t_k - \epsilon) \qquad \text{and} \qquad \lim_{\epsilon = 0} F(t_k + \epsilon)$$

become one $+\infty$, *the other* $-\infty$. *Assume furthermore that* u *within the period-interval vanishes for a finite number* μ *of values of* t. *Under these conditions* $\nu + \mu$ *is an even number. When* $\nu = 0$, *i. e. when* u *does not become infinite within the period-interval, then* μ, *the number of roots, when finite, is even.*

There is no loss in generality in the assumption of single zeros for u, and $t_k \neq 0$, $k = 1, 2, 3, \cdots, \nu$.

THEOREM 2. *Assume again a function* $G(t)$ *as defined above, but without infinities within the period-interval; moreover that the derivatives of* $G(t)$ *are functions of the same type as* $G(t)$. *Then every derivative of* $G(t)$ *is coperiodic with* $G(t)$ *and has at least two, and generally an even number of roots.*

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2. Definition of a Closed Curve. Any closed curve may be represented parametrically in the form

$$x = \phi(t), \quad y = \psi(t), \tag{1}$$

where $\phi(t)$ and $\psi(t)$ are uniform, continuous real functions for all values of the real parameter t, and with the same period w.* It is moreover assumed that the curve has a continuous and uniform tangent for a continuous and uniform change of the parameter. The system of Cartesian coördinates (x, y) may always be chosen in such a manner, that the tangent at no singular point becomes parallel to either coördinate axis, and that $\phi(t)$ and $\psi(t)$ and all their derivatives are functions with the properties stated in Theorem 2. In the definition of the closed curve (1) we exclude rectilinear segments, and admit only arcs with singularities of an algebraic type, which shall be finite in number. Such a curve is called an *ordinary closed curve.*

3. Tangents From a Point to an Ordinary Closed Curve. Take any point (x_1, y_1) in the plane of the curve as defined, and not on the curve. The equation of the tangent at the point (x, y) of the curve is

$$\eta - y = \frac{dy}{dx}(\xi - x),$$

ξ and η denoting current coördinates. The condition that the tangent shall pass through (x_1, y_1) is, by substituting for x and y their parametric expressions and reducing,

$$[\phi(t) - x_1]\frac{d\psi}{dt} - [\psi(t) - y_1]\frac{d\phi}{dt} = 0. \tag{2}$$

The left-hand member of this equation is evidently a function of the type $G(t)$ as defined in Theorem 2, and vanishes therefore for no value or an even number of values of t within the period-interval. It is known that the values of t for which $\phi'(t)$ and $\psi'(t)$ vanish simultaneously define the cusps of the curve. These are therefore contained among the roots of (2). All the other roots determine proper tangents. A tangent at a point determined by a multiple root must be counted as often as the degree of multiplicity indicates. Thus, counting multiplicities properly, we have

THEOREM 3. *The number of tangents from a fixed point to an ordinary closed curve and its cusps is even, including, by definition, zero as an even number.*

When there are no cusps, then the number of tangents is even. When (x_1, y_1) is on the curve, the tangent at this point must be counted twice;

* See W. F. Osgood, Lehrbuch der Funktionentheorie, vol. I, 2d edition, pp. 140–150.

likewise, when the tangent through (x_1, y_1) is an inflectional tangent. A proper double-tangent through (x_1, y_1) is determined by two distinct values t_1 and t_2, and must, of course, be counted twice.

4. Inflexions and Cusps of Ordinary Closed Curves. Consider the function of t, formed from the parametric expressions of the curve (1),

$$W = \frac{u}{v},$$

where

$$u = \frac{d^2y}{dt^2} - \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}, \qquad v = \left(\frac{dx}{dt}\right)^2.$$

The curve has inflexions for those values of t for which W vanishes. It is evident that u, v, W are all coperiodic with $\phi(t)$ and $\psi(t)$. As the latter and all their derivatives remain finite for all values of t, u can become infinite only when dy/dx becomes infinite. This will be the case when $dx/dt = 0$, $dy/dt \neq 0$, simultaneously. Hence, for such values $t = t_\tau$, for which this is the case, $|u| = +\infty$, and the curve has tangents parallel to the y-axis. In the neighborhood of such a point, as t increases continuously from $t_\tau - \epsilon$ to $t_\tau + \epsilon$, dy/dx changes abruptly from $\pm\infty$ to $\mp\infty$, while d^2x/dt^2 and d^2y/dt^2 remain finite. For all other values of t, different from t_τ, u is finite and continuous. u is therefore exactly a function as defined under Theorem 1, and admits therefore of an even number

$$2k = \mu + \nu$$

of zeros and infinities within the period interval, as specified. Points where we have $W = \infty$, and $dy/dx \neq \infty$ are defined as cusps. These are distinguished as cusps of the first kind, in the neighborhood of which d^2y/dx^2 changes signs; and cusps of the second kind, in the neighborhood of which d^2y/dx^2 does not change sign. As v is always positive, u, which remains finite, must change signs in case of a cusp of the first kind. In other words the cusps of the first kind are determined by a part c_1 of the number of roots of $u = 0$. Thus, the condition for a cusp of the first kind may be briefly stated by $u = 0$, $dx/dt = 0$. The remaining number i of roots of $u = 0$ determines the number of inflexions ($u = 0$, $dx/dt \neq 0$). The number of roots of u is therefore made up of c_1 and i, or $\mu = i + c_1$, so that according to Theorem 1

$$2k = i + c_1 + \nu. \tag{3}$$

Hence

THEOREM 4. *The number of tangents parallel to a given direction, inflexions and cusps of the first kind of an ordinary closed curve is even.*

When such tangents occur at cusps of the second kind, they must be included in this number.

In case of a cusp of the second kind, $u \neq 0$ and finite, and $dx/dt = 0$. The number c_2 of cusps of the second kind is contained in the number $2n$ of roots of $v = 0$, or $dx/dt = 0$, which, since $\phi'(t)$ is a periodic function as described under Theorem 2, is even. The number $2n$ is composed of the number ν of tangents parallel to the y-axis, the numbers c_1 and c_2 of cusps of the first and second kind; i. e.,

$$2n = c_1 + c_2 + \nu. \tag{4}$$

From (4) and (3) follows

$$i + c_2 = 2(n + k - \nu - c_1), \tag{5}$$

which is an even number. Hence

THEOREM 5. *An ordinary closed curve has an even number of inflexions and cusps of the second kind.*

Subtracting (4) from (3), we get

$$i - c_2 = 2(k - n), \tag{6}$$

an even number. Hence

THEOREM 6. *The difference between the number of inflexions and the number of cusps of the second kind of an ordinary closed curve is even.*

5. Algebraic Curves. When the closed curve is a complete irreducible algebraic curve, then its order is even and the theorem on inflexions follows immediately from Plücker's formulas. Denoting by 2α, N, i, d, r, order, class, number of inflexions (including cusps of the second kind), double-points, cusps, we have

$$i = 6\alpha(2\alpha - 2) - 6d - 8r,$$

an even number of inflexions. For the class N we have

$$N = 2\alpha(2\alpha - 1) - 2d - 3r,$$

and

$$N + r = 2[\alpha(2\alpha - 1) - d - r],$$

which is even. Hence as found before, the number of cusps and tangents from a point to the curve is even.*

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* For other topological properties of closed curves see G. Landsberg: "Beitrag zur Topologie geschlossener Kurven mit Knotenpunkten und zur Kroneckerschen Charakteristikentheorie," Math. Annalen, vol. 70 (1911), pp. 563–579; in the same publication A. Kneser: "Einige allgemeine Sätze über die einfachsten Gestalten ebener Kurven," vol. 41 (1893), pp. 347–376. v. Staudt in his Geometrie der Lage, p. 113 (1847), gave a purely intuitional proof of the theorem that every closed curve has an even number of inflexions (defined in the sense of v. Staudt).